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NEIGHBORLY 6-POLYTOPES WITH 10 VERTICES

BY

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ABSTRACT

There are exactly 37 combinatorial types of neighborly 6-polytopes with 10 vertices. A full description is given.

1. Introduction

A *neighborly* d-polytope is a d-polytope K (in $\mathbf{R}^d$) such that the convex hull of any $[\frac{1}{2}d]$ vertices of K is a face of K. A well-known family of simplicial neighborly polytopes is the class of cyclic polytopes. (Cf. Grünbaum [GR] section 4.7, chapter 7 and section 9.6 for the basic facts concerning neighborly and cyclic polytopes.)

Denote by $g(v,d)$ the number of combinatorial types of neighborly d-polytopes with v vertices, and by $g_s(v,d)$ the number of combinatorial types of simplicial neighborly d-polytopes with v vertices.

Note that for even d, $g_s(v,d)=g(v,d)$. For every d and $v>d$ there is a cyclic d-polytope with v vertices, hence $g_s(v,d)\geqq 1$. Barnette [BR] and the second author [SH2] proved independently that $g_s(v,d)\rightarrow\infty$ as $v\rightarrow\infty$, for any fixed $d\geqq 3$. For even d, $g(d+1,d)=g(d+2,d)=g(d+3,d)=1$. For odd $d\geqq 3$, $g(d+1,d)=g_s(d+1,d)=g_s(d+2,d)=1$, and $g(d+2,d)=2$. Altshuler and McMullen [AM] computed $g_s(d+3,d)$ for odd d. The first interesting case in even dimension is $d=4$, $v=8$. Grünbaum [GR, p. 124] showed that $g(8,4)>1$ and, in fact, $g(8,4)=3$ (Grünbaum and Sreedharan [GS]). The second author showed that

$$g(2m+4,2m)>\frac{1}{4(m+2)}\prod_{i=2}^{m}(2i+1)$$

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(see [SH2]). It is also known that $g(9,4) = 23$ and $g(10,4) = 431$ (see [SH1], [AS], [AL], [BoSt2]). In this paper we deal with the first non-trivial case in dimension 6, i.e., $v = 10$.

The main result of this paper is:

THEOREM 1. *There are exactly* 37 *combinatorial types of neighborly* 6-*polytopes with* 10 *vertices.*

The problems of enumerating d-polytopes and combinatorial $(d-1)$-spheres are closely related. Although we tried to avoid generating an excessive number of non-polytopal spheres, we still obtained as a by-product of our computations:

THEOREM 2. *There are at least* 14 *non-polytopal combinatorial types of* 3-*neighborly simplicial* 5-*spheres with* 10 *vertices.*

In Section 2 we survey the concepts and results needed for enumerating neighborly 6-polytopes with 10 vertices. In Section 3 we describe the methods used in the computation, and we prove that $g(10,6) \leqq 37$. In Sections 4, 5 and 6 we deal with the realization problem and show that $g(10,6) = 37$.

We use the notation of [GR].

2. Theoretical background

We obtained the polytopes with 10 vertices by adding a tenth vertex to a polytope with 9 vertices, using the beneath-beyond technique (see Grünbaum [GR, section 5.2]). We need some results about the connection between the facial structure of a neighborly polytope and the facial structure of its subpolytopes.

Throughout this section, the letter Q denotes a neighborly $2m$-polytope, not a simplex. Since Q is a simplicial polytope, we can identify each proper face of Q with its set of vertices. Thus, we regard the boundary complex $\mathscr{B}(Q)$ as an abstract simplicial complex (i.e., a collection of sets, closed under the operation of taking a subset).

A set M of vertices of Q is a *missing face* of Q if $M \notin \mathscr{B}(Q)$ but $S \in \mathscr{B}(Q)$ for every $S \subsetneqq M$.

We denote by $\mathscr{M}(Q)$ the set of missing faces of Q. The following Lemma is trivial (see Altshuler and Perles [AP, section 2]):

LEMMA 1. *A subset T of* vert Q *belongs to* $\mathscr{B}(Q)$ *iff no subset of T belongs to* $\mathscr{M}(Q)$. ∎

The second author [SH2, theorem 2.4] proved that all missing faces of Q are of size $m+1$. This implies:

LEMMA 2. *Let M be a subset of* vert Q. *M is a missing face of Q iff* $|M| = m+1$ *and* $M \notin \mathcal{B}(Q)$.

Choose a vertex x of Q and define: $V = \text{vert}\, Q \setminus \{x\}$, $P = \text{conv}\, V$. For a facet F of P, we say that *x covers F* if x lies beyond F with respect to P. Denote by $\mathcal{C}$ the set of facets of P that x covers. Note that x lies beneath all the facets of P not in $\mathcal{C}$. $\mathcal{B}(Q)$ is determined by $\mathcal{B}(P)$ and $\mathcal{C}$, as follows (cf. Grünbaum [GR, section 5.2]):

LEMMA 3. *Let T be a subset of V. $T \in \mathcal{B}(Q)$ iff P has a facet F which includes T and $F \notin \mathcal{C}$. $T \cup \{x\} \in \mathcal{B}(Q)$ iff P has facets F_1, F_2 which include T, $F_1 \in \mathcal{C}$ and $F_2 \notin \mathcal{C}$.* ∎

The second author proved [SH2, lemma 2.11] that $\mathcal{M}(Q)$ and x determine $\mathcal{M}(P)$:

LEMMA 4. *Assume $M \subset V$, $|M| = m+1$. Then $M \in \mathcal{M}(P)$ iff $M \in \mathcal{M}(Q)$ and $(M \setminus \{t\}) \cup \{x\} \in \mathcal{M}(Q)$ for some t in M.* ∎

Let Δ be an abstract simplicial complex. A face (i.e., member) F of Δ is *j-dimensional* (or a *j-face*) if $|F| = j+1$. Δ is a *d-complex* (or a *d-dimensional complex*) if it has a d-face but no $(d+1)$-face. Δ is *homogeneous* if all its maximal faces have the same dimension.

For a finite set F we denote the set of all subsets of F by $\bar{F}$. For a collection $\mathcal{D}$ of finite sets we denote by $\Delta(\mathcal{D})$ the complex $\bigcup_{F \in \mathcal{D}} \bar{F}$.

Let $\mathcal{F}$ be a collection of facets of a simplicial $(d+1)$-polytope P. $\mathcal{F}$ generates a d-dimensional subcomplex $\Delta = \Delta(\mathcal{F})$ of $\mathcal{B}(P)$. Define the *boundary complex* $\partial\Delta$ as follows: The maximal faces of $\partial\Delta$ are the subfacets of P that are contained in exactly one member of $\mathcal{F}$. Each F in $\partial\Delta$ is called a *boundary face* of Δ; the remaining faces of Δ are its *interior faces.*

It turns out that Φ is an interior face of Δ iff all facets of P that include Φ belong to $\mathcal{F}$. This follows, by duality, from the connectivity of the graph of the face $\hat{\Phi}$ dual to Φ in the polytope P^* dual to P.

Define: $\Delta^\circ = \Delta \setminus \partial\Delta$. For $-1 \leqq j \leqq d$, $f_j(\Delta)$, $f_j(\partial\Delta)$ and $f_j(\Delta^\circ)$ denote the number of j-faces of P which belong to Δ, $\partial\Delta$ and Δ° respectively.

Define: $f(\Delta) = (f_0(\Delta), \ldots, f_d(\Delta))$. $f(\partial\Delta)$, $f(\Delta^\circ)$ are defined similarly.

With the f-vector $f(\Delta)$ of Δ we associate an h-vector $h(\Delta) = (h_0, \ldots, h_{d+1})$, defined by:

$$h_j = h_j(\Delta) = \sum_{i=0}^{j} (-1)^{i-j} \binom{d+1-i}{d+1-j} f_{i-1}(\Delta), \qquad 0 \leqq j \leqq d+1.$$

Now we are ready to discuss the shellability of $\mathscr{C}$. Several equivalent definitions of this notion have appeared in the literature ([DK1], [DK2], [BM], [BL]). We chose the following definition ([BL]):

An abstract simplicial d-complex Δ is *shellable* if Δ is homogeneous, and there is an ordering $F_1, F_2, \ldots, F_w$ of its maximal faces such that for every k, $2 \leqq k \leqq w$,

$$(*) \qquad \bar{F}_k \cap \left(\bigcup_{j=1}^{k-1} \bar{F}_j \right) = \bigcup_{j=1}^{s_k} \bar{G}_j^k,$$

where $1 \leqq s_k \leqq d+1$, and $G_1^k, \ldots, G_{s_k}^k$ are distinct d-subsets of F_k.

The sequence $F_1, \ldots, F_w$ is called a *shelling* of Δ.

Remark: If every $(d-1)$-face of Δ is included in at most two d-faces of Δ, then s_k is just the number of F_j's, $j < k$, that are adjacent to F_k (i.e., $|F_j \cap F_k| = d$). It is known that $h_j(\Delta) = |\{k : 2 \leqq k \leqq w, s_k = j\}|$, $1 \leqq j \leqq d+1$. (See [BL, proposition 2], [MS, section 5.2].)

Condition (*) is equivalent to the following condition, which is handier in computations:

If $1 \leqq j < k$ and $|F_k \cap F_j| < d$, then $F_k \cap F_j \subset F_i$ and $|F_k \cap F_i| = d$ for some $1 \leqq i < k$.

A non-empty finite collection $\mathscr{F}$ of $(d+1)$-sets is *shellable* if the d-complex $\Delta(\mathscr{F})$ is shellable.

Lemma 5. *Let $\mathscr{F}$ be a collection of facets of a simplicial $(d+1)$-polytope. Assume $F_1, \ldots, F_w$ is a shelling of $\mathscr{F}$. Put $\Delta = \Delta(\mathscr{F})$. If $\Phi \in \Delta^\circ$, then $\Phi = F_1$ or $\Phi \supset \bigcap_{i=1}^{s_k} G_i^k$ for some $2 \leqq k \leqq w$.*

Proof. Assume $\Phi \in \Delta^\circ$ and $\Phi \neq F_1$. Then $\Phi \subset F_j$ for some $2 \leqq j \leqq w$. Let k be the last index j such that $\Phi \subset F_j$. If Φ does not include $\bigcap_{i=1}^{s_k} G_i^k$, then F_k has a d-subset G that includes Φ but not $\bigcap_{i=1}^{s_k} G_i^k$. Clearly, G is not a subset of F_j for $j \neq k$, thus $G \in \partial\Delta$, contradicting our assumption that $\Phi \in \Delta^\circ$. ■

It follows that, under the assumptions of Lemma 5,

$$f_j(\Delta^\circ) \leqq \sum_{i=d-j}^{d+1} \binom{i}{d-j} h_i, \qquad 0 \leqq j \leqq d.$$

Remark. The converse of Lemma 5 is also true.

By the work of Bruggesser and Mani [BM], the set $\mathscr{C}$ of facets of P covered by

x is shellable. Using the methods of Danaraj and Klee [DK1], it is easy to prove the following lemma:

LEMMA 6. *Let Φ be a face of P. Assume all the facets of P that include Φ, t in number, are in $\mathscr{C}$. Then $\mathscr{C}$ has a shelling $F_1, \ldots, F_t, \ldots, F_w$, such that $\Phi \subset F_j$ for $1 \leqq j \leqq t$.* ■

Define: $\Delta = \Delta(\mathscr{C})$. Lemma 3 implies the following:

LEMMA 7. $\operatorname{link}(x; \mathscr{B}(Q)) = \partial \Delta$ *and* $\operatorname{ast}(x; \mathscr{B}(Q)) = \mathscr{B}(P) \setminus \Delta^\circ$. ■

Let H be a hyperplane that strictly separates x from P. The polytope $Q_x = Q \cap H$ is a vertex figure of Q at x. It is well known that the complexes $\mathscr{B}(Q_x)$ and $\operatorname{link}(x; \mathscr{B}(Q))$ are isomorphic. Hence $f_j(\partial\Delta) = f_j(Q_x)$ for $0 \leqq j \leqq 2m-2$. Therefore

$$\begin{aligned} f_j(\Delta^\circ) &= f_j(P) - f_j(\operatorname{ast}(x; \mathscr{B}(Q))) \\ &= f_j(P) - f_j(Q) + f_{j-1}(\operatorname{link}(x; \mathscr{B}(Q))) \\ &= f_j(P) - f_j(Q) + f_{j-1}(Q_x), \qquad 0 \leqq j \leqq 2m-1. \end{aligned}$$

The f-vector $f(K)$ of a simplicial neighborly d-polytope K is a function of d and $|\operatorname{vert} K|$ only. Since Q_x is a neighborly $(2m-1)$-polytope, it follows that:

LEMMA 8. *$f(\Delta^\circ)$, $f(\partial\Delta)$, $f(\Delta)$ and $h(\Delta)$ are functions of $|V|$ and m only.* ■

Assume in the sequel that $m = 3$ and $|V| = 9$. Then

$$f(Q) = (10, 45, 120, 185, 150, 50), \quad f(P) = (9, 36, 84, 117, 90, 30),$$
$$f(Q_x) = (9, 36, 74, 75, 30).$$

(See [GR, Table 3, p. 425].) The formulas given above for $f_j(\Delta^\circ)$ and $h(\Delta)$ yield:

LEMMA 9. $f(\Delta^\circ) = (0, 0, 0, 6, 15, 10)$, $f(\Delta) = (9, 36, 74, 81, 45, 10)$ *and* $h(\Delta) = (1, 3, 6, 0, 0, 0, 0)$. ■

This implies, by Lemma 5 and the Remark preceding it:

LEMMA 10. *Let $F_1, \ldots, F_{10}$ be a shelling of $\mathscr{C}$. Then six facets F_k are adjacent to exactly two members of $\{F_1, \ldots, F_{k-1}\}$, and three facets F_k are adjacent to exactly one member of $\{F_1, \ldots, F_{k-1}\}$. The six 3-faces of Δ° are exactly the intersections $G_1^k \cap G_2^k$ $(2 \leqq k \leqq 10,\ s_k = 2)$.* ■

Q has 25 missing faces, since

$$\binom{10}{4} - f_3(Q) = 210 - 185 = 25.$$

P has 9 missing faces. By Lemmas 2 and 3, the six 3-faces of Δ° are missing faces of Q. Therefore Q has $25-9-6=10$ missing faces that contain the vertex x. Since x is an arbitrary vertex of Q, we conclude that every vertex of Q is contained in exactly 10 missing faces of Q.

A useful tool for classifying finite combinatorial structures is the *edge-valence matrix* (cf. [AS]) defined below.

For every two vertices (not necessarily different) x, y of Q, define:

$$\mathrm{EV}(Q,x,y)=|\{M\in\mathcal{M}(Q)\colon \{x,y\}\subset M\}|.$$

Choose an arbitrary ordering $x_1,\dots,x_{10}$ of vert Q. Define an edge-valence matrix $\mathrm{EV}(Q)=(a_{ij})$ by $a_{ij}=\mathrm{EV}(Q,x_i,x_j)$, $1\leqq i,\ j\leqq 10$.

By the remark above, $a_{ii}=10$ for all i.

The matrix $\mathrm{EV}(Q)$ is symmetric. A different ordering of vert Q would yield another matrix $\mathrm{EV}(Q)$, similar to the original one. Thus

LEMMA 11. det $\mathrm{EV}(Q)$ *is an invariant of the combinatorial type of* Q. ■

REMARKS. Lemma 11 remains true if the diagonal entries a_{ii} are replaced by $\beta(\mathrm{EV}(Q,x_i,x_i))$ and the non-diagonal entries a_{ij} are replaced by $\gamma(\mathrm{EV}(Q,x_i,x_j))$ where β and γ are arbitrary real valued functions, defined on the natural numbers.

The set $\mathcal{M}(Q)$ in the definition of $\mathrm{EV}(Q,x,y)$ might as well be replaced by the set of all facets of Q, as in [AS]. Obviously, Lemma 11 holds also for this version of $\mathrm{EV}(Q)$. We shall use this "facet-edge-valence matrix" in step 3 of the computations below.

An edge ab of Q is *universal* if for every $(m-1)$-subset of vert Q, $\{a,b\}\cup S\in\mathcal{B}(Q)$ (or, equivalently, $\{a,b\}\cup S\notin\mathcal{M}(Q)$).

LEMMA 12. *A neighborly* $2m$*-polytope with* $2m+3$ *or more vertices is cyclic iff it has a hamiltonian circuit of universal edges.*

PROOF. See [SH2, theorem 3.5]. ■

An edge ab of Q is universal iff $\mathrm{EV}(Q,a,b)=0$. Therefore one can easily see from $\mathrm{EV}(Q)$ if Q is cyclic or not.

3. Computations

From here on P denotes a cyclic 6-polytope with 9 vertices. Denote the vertices of P by $1,2,\dots,9$, with the cyclic order $1,2,\dots,9,1$.

The facets of P are:

123456	123467	124567	234567	123478
124578	234578	125678	235678	345678
123459	123569	134569	123679	134679
145679	123489	124589	234589	125689
235689	345689	123789	134789	145789
126789	236789	346789	156789	456789

The missing faces of P are:

1357 1358 1368 1468 2468 2469 2479 2579 3579

Step 1. In this step we found all collections $\mathscr{C}$ of 10 facets of P which satisfy the following conditions:

(1) $\mathscr{C}$ has a shelling $F_1, \ldots, F_{10}$.

(2) Six facets F_k are adjacent to exactly two members of $\{F_1, \ldots, F_{k-1}\}$ and three facets F_k are adjacent to exactly one member of $\{F_1, \ldots, F_{k-1}\}$.

(3) For every interior 3-face Φ of $\Delta(\mathscr{C})$, $\mathscr{C}$ has a shelling $F_1, \ldots, F_{10}$, such that $\Phi \subset F_j$ for $1 \leqq j \leqq t$, where $t = |\{F \in \mathscr{C}: F \supset \Phi\}|$.

By Lemmas 6 and 10, these conditions are necessary for $\mathscr{C}$ to be a "covered cap".

In step 2 we associated with each collection $\mathscr{C}$ a complex (see (**) below), which may (or may not) be isomorphic to the boundary complex of a neighborly 6-polytope with 10 vertices.

If an automorphism φ of P maps $\mathscr{C}_1$ onto $\mathscr{C}_2$, then the corresponding complexes are, clearly, isomorphic by the mapping $i \to \varphi(i)$ for $1 \leqq i \leqq 9$ and $10 \to 10$. Hence it suffices to take a representative of each equivalence class of collections $\mathscr{C}$.

It seems worthwhile to describe, in some detail, our algorithm for finding all those collections $\mathscr{C}$.

Under the automorphism group aut P of P there are 9 equivalence classes of 3-faces of P, with representatives $\Phi_1, \ldots, \Phi_9$:

1234 1245 1256 1236 1247 1346 1246 1235 1458

We divided the collections $\mathscr{C}$ into 9 classes as follows: $\mathscr{C}$ belongs to class i ($1 \leqq i \leqq 9$) if the complex $\Delta(\mathscr{C})$ has an interior 3-face equivalent (under aut P) to Φ_i, but has no interior 3-face equivalent to Φ_j ($1 \leqq j < i$).

Call a collection $\mathscr{C}$ of class i *special* if Φ_i itself is an interior face of $\Delta(\mathscr{C})$. Clearly every collection of class i is equivalent under aut P to a special one.

To generate the special collections of class 1, consider Φ_1. Φ_1 is included in 5 facets $F_1, \dots, F_5$ of P:

$$123456 \quad 123467 \quad 123478 \quad 123489 \quad 123459.$$

Note that $F_1, \dots, F_5$ is a shelling, and the sequence (s_2, s_3, s_4, s_5) is $(1,1,1,2)$. We extended the shelling $F_1, \dots, F_5$ by a backtrack algorithm (see Danaraj and Klee [DK2]), in all possible ways, to a shelling $F_1, \dots, F_{10}$, with the restriction that $s_k = 2$ for $6 \leqq k \leqq 10$. This procedure clearly yields all the special collections $\mathscr{C}$ of class 1.

To generate the special collections of class 2 we started with the facets $F_1, \dots, F_5$ that include Φ_2:

$$123456 \quad 124567 \quad 124578 \quad 124589 \quad 123459.$$

As before, we extended $F_1, \dots, F_5$ to a shelling $F_1, \dots, F_{10}$, with $s_k = 2$ for $6 \leqq k \leqq 10$.

Assume the facet F is a candidate for being chosen as F_k, for some $6 \leqq k \leqq 10$; that is, $\bar{F} \cap (\bigcup_{j=1}^{k-1} \bar{F}_j) = \bar{G}_1 \cup \bar{G}_2$, where G_1, G_2 are two distinct 5-subsets of F. If the 3-face $\Phi = G_1 \cap G_2$ is equivalent to Φ_1, then F is ruled out, since the choice $F_k = F$ would lead to a collection $\mathscr{C}$ of class 1.

In the same manner we generated the special collections of class i, $i = 3, \dots, 9$, ruling out interior 3-faces that are equivalent to Φ_j for some $j < i$.

We obtained altogether 371 non-equivalent collections $\mathscr{C}$, as follows:

Class	1	2	3	4	5	6	7	8	9
Number of collections	45	75	79	87	84	1	0	0	0

Since we did not check whether condition (3) holds for each interior 3-face of $\Delta(\mathscr{C})$, it is conceivable that some of the 371 collections obtained fail to satisfy condition (3).

Step 2. For every collection $\mathscr{C}$ found in Step 1 we constructed the complex

$$(**) \qquad \mathscr{B}(Q) = (\mathscr{B}(P) \backslash \Delta(\mathscr{C})^\circ) \cup \{F \cup \{10\}: F \in \partial\Delta(\mathscr{C})\}.$$

Note that if $\mathscr{C}$ is a "covered cap", i.e., if $\mathscr{C}$ is the set of facets of P covered by some point x (for some realization of P), then $\mathscr{B}(Q)$ is just the boundary complex of the polytope $Q = \text{conv}(P \cup \{x\})$ with x replaced by 10. (See Lemma 7.) If $\mathscr{C}$ is not a "covered cap", then $\mathscr{B}(Q)$ is still a (triangulation of a) 5-sphere.

This follows from the shellability of $\mathscr{C}$. We omit the details, since we shall never use this fact, except in the statement of Theorem 2. For each complex $\mathscr{B}(Q)$ we calculated its set of missing faces, denoted by $\mathscr{M}(Q)$. As expected, all the missing faces found were of size 4.

Then we sorted the complexes $\mathscr{B}(Q)$ according to the determinant $\det(Q) = \det \mathrm{EV}(Q)$.

If $\det(Q) \neq \det(Q')$, then $\mathscr{B}(Q)$ and $\mathscr{B}(Q')$ are not isomorphic. We checked and found that $\mathscr{B}(Q)$ and $\mathscr{B}(Q')$ were isomorphic whenever $\det(Q) = \det(Q')$.

We obtained 51 equivalence classes of spheres. These are listed in Table 1, in terms of their missing faces, in increasing order of their determinant. The vertex number 10 is represented as 0 in Tables 1, 2, 3.

The complexes listed in Table 1 were obtained from the complexes $\mathscr{B}(Q)$ described above by renaming the vertices. After renaming, each equivalence class of vert Q_i with respect to aut Q_i became a set of consecutive numbers.

Step 3. In this step we proved, in three ways, that 14 spheres are not polytopal.

Let Q be one of the 51 spheres listed in Table 1, and let x be a vertex of Q. We say that *Q is obtained at x* if there is a sphere Q' among the 371 spheres considered in step 2, and a combinatorial equivalence φ from $\mathscr{B}(Q')$ onto $\mathscr{B}(Q)$, such that $x = \varphi(10)$.

Note that if Q is a neighborly 6-polytope with 10 vertices, and $x \in \operatorname{vert} Q$, then $\operatorname{conv}(\operatorname{vert} Q \setminus \{x\})$ is cyclic, i.e., combinatorially equivalent to P.

Since we have chosen all possible "covered caps" $\mathscr{C}$, up to equivalence under aut P, we have:

LEMMA 13. *A polytopal sphere is obtained at all its vertices.* ∎

14 spheres in our list (Q_{38}–Q_{51}) were not obtained at all their vertices, thus establishing:

$$g(10,6) \leqq 37.$$

Now we show two additional proofs of the non-polytopality of Q_{38}–Q_{51}. Suppose Q_i is a polytope.

Second proof: For a vertex x of Q_i, find the missing faces of the subpolytope $Q_i(x) = \operatorname{conv}(\operatorname{vert} Q_i \setminus \{x\})$ according to the rule of Lemma 4. Since $Q_i(x)$ is combinatorially equivalent to P, we should obtain 9 missing faces, for every choice of x. But in each of the cases $38 \leqq i \leqq 51$ we obtained 11 missing faces, for at least one choice of x.

TABLE 1
Representatives of classes

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_1	1357	1358	1368	1468	2468	D_{20}	
	1359	1369	1469	2469	1379		1 2 3 4 5
125000	1479	2479	1579	2579	3579	(1,2,3,4,5,6,7,8,9,0)	
	2460	2470	2570	3570	2480	(1,9)(2,8)(3,7)(4,6)	0 9 8 7 6
2	2580	3580	3680	3680	4680		
Q_2	2357	2457	1368	1468	1378	Z_2	
	2378	1478	2478	2359	1369		7 6 2 1 5 8
165000	3569	2579	1389	2389	3689	(1,2)(3,4)(5,6)(7,8)(9,0)	
	2450	1460	4560	1470	2470		9 4 3 0
2	4570	1680	2590	1690	5690		
Q_3	1267	1467	1238	2358	1268	e	
	2568	2678	2359	3459	2579		8 4 2 0
174720	4579	2679	4679	2589	3589		
	1340	3450	1460	1470	1670		6 3 7; 9 1 5
2	1380	3580	1680	3490	4790		
Q_4	2367	1348	1458	2368	3468	e	
	2568	4568	2678	1459	1579		7 4 2 1 6
176520	2579	2679	1589	2589	4589		
	1340	1450	2360	3460	1370		9 3 5; 8 0
2	2370	3670	1490	1790	2790		
Q_5	2357	1468	3578	4678	5678	Z_2	
	1239	1249	2359	2459	1469		5 1 7; 6 2 8
187392	2579	1489	4589	4689	5789	(1,2)(3,4)(5,6)(7,8)(9,0)	
	1230	1240	2350	1360	1460		3 4; 9 0
2	2370	3570	3670	1680	6780		
Q_6	2357	1457	2457	1368	2368	Z_2	
	1468	1378	2378	1478	2478		1 2; 3 4
194112	2359	2369	2379	2579	3689	(1,2)(3,4)(5,6)(7,8)(9,0)	
	1450	1460	4570	1480	1680		5 8; 6 7
4	2590	4590	1690	3690	5690		
Q_7	3456	2357	3457	3567	1468	Z_2	
	3468	4568	2359	3569	1279		0 3 1 5
197960	2379	1289	2389	1689	3689	(1,2)(3,4)(5,6)(7,8)(9,0)	
	1460	4560	1270	1470	2570		9 4 2 6; 7 8
2	4570	1280	1480	2790	1890		
Q_8	2345	3456	2457	3458	1368	Z_2	
	3468	2459	1279	2579	4589		4 1 5
199680	1689	4689	1789	5789	6789	(2,3)(4,5)(6,7)(8,9)	
	1230	2340	2350	1360	3460		6 2 8; 7 3 9
2	1270	2570	1670	1680	1790		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_9	2345	3456	2457	4567	1368	D_6	
	3568	4568	1279	2479	4579		4 1 5
204800	1689	5689	1789	4789	6789	(1, 2, 3)(4, 6, 9)(5, 8, 7)	
	1230	2340	2350	1360	3560	(1, 2)(4, 8)(5, 6)(7, 9)	9 3 7; 6 2 8
3	1270	2470	1380	1290	1890		
Q_{10}	2456	2357	2567	1468	4568	$Z_2 + Z_2$	
	1378	3578	1678	5678	2359		8 2 1 5
205000	2459	2569	1379	2379	3789	(1, 2)(3, 4)(5, 8)(6, 7)(9, 0)	
	1460	2460	4560	1380	1480	(1, 3)(2, 4)(5, 6)(7, 8)(9, 0)	7 4 3 6
2	1780	1390	2390	1490	2490		
Q_{11}	1256	2356	1456	1257	2357	Z_2	
	1268	1468	1278	2378	1478		3 1 9; 6 7
206336	3478	2359	2569	2379	3789	(1, 2)(3, 4)(5, 6)(7, 8)(9, 0)	
	1460	1560	1480	4780	3490		4 2 0; 5 8
2	3590	4690	5690	3790	4890		
Q_{12}	1256	2356	1456	2357	2567	Z_2	
	1468	1568	1269	1469	2379	(1, 2)(3, 4)(5, 6)(7, 8)(0, 0)	3 1 7; 5 9
206448	3479	2679	4679	1489	4789		
	1250	2350	2370	1480	3480		4 2 8; 6 0
2	1580	3580	3780	3790	4890		
Q_{13}	3456	2357	3457	3567	2358	e	
	3458	1468	3468	1469	4569		5 1 3 9
206600	1279	2579	1679	5679	4689		
	1270	2370	2570	1280	2380		4 2 6; 7 8
2	1480	3480	1290	1690	1890		
Q_{14}	1356	1456	1367	1467	2358	e	
	3568	2378	2478	3678	4678		6 2 1 8
207240	1469	1569	1479	2479	4789		
	1350	2350	3560	2380	2780		4 3 9; 5 7
2	1490	2490	1590	2590	2890		
Q_{15}	2357	3457	1467	3467	3567	Z_2	
	2358	3458	1468	3466	4568	(1, 2)(3, 4)(5, 6)(7, 8)(9, 0)	3 1 5
207368	2359	1289	1489	2589	4589		
	1460	1270	2370	1670	3670		4 2 6; 7 8
3	1290	2590	1690	2790	1890		
Q_{16}	2347	2457	3467	4567	1258	Z_3	
	2458	1568	4568	2478	1369	(1, 2, 3)(4, 6, 5)(7, 9, 8)	4 1 7
209088	3469	1569	4569	3679	1589		
	1230	2370	3470	1280	2580		6 2 9; 5 3 8
2	2780	1390	1690	3790	1890		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_{17}	2345	3456	2347	2348	3458	e	
	1278	2378	1569	4569	3489		3–1–4 5–7
209568	1689	4689	1789	3789	6789		
	1250	2350	2450	1560	4560		6–2–9 8–0
2	1270	2370	1670	1690	1790		
Q_{18}	2345	3456	1248	2348	2458	e	
	1278	3459	3569	3579	1679		3–1–5 2–9
212272	3679	1489	3489	1789	3789		
	1240	2450	1260	2560	3560		4–7 6–8
2	1670	5670	1280	1780	6790		
Q_{19}	2457	1368	4578	1678	4678	e	
	5678	1239	2349	2459	1369		4–1–5
213208	1279	2479	1389	1789	4789		
	1230	2350	2450	1360	3560		6–2–8 3–7
2	4560	4570	1680	5680	2390		
Q_{20}	2456	2458	1368	2368	2568	e	
	1378	2378	1478	2478	2578		2–1–5 3–4
215800	2459	1379	1479	2479	4579		
	1360	2560	3560	1370	3680		6–7 8–9
2	1390	1490	4590	1690	5690		
Q_{21}	2356	3567	3568	1478	3578	e	
	4578	1249	2349	2359	1269		3–1–5
217032	2369	1479	3479	3579	4789		
	1240	1260	2360	2560	1480		7–2–8 4–6
2	1680	5680	1780	5780	1490		
Q_{22}	1467	2467	2348	2358	2458	e	
	2468	2378	2678	1359	1379		2–1–8 3–6
217408	2379	1679	2679	3589	3789		
	1350	1450	1460	2460	4560		4–9 5–7
2	3580	4580	1590	1690	1790		
Q_{23}	2356	1457	3457	3467	3567	e	
	2368	3468	3678	1479	3479		3–1–6
216624	1289	1489	2689	4689	4789		
	1250	2350	2560	1570	3570		4–2–7 5–8
2	1280	2680	1290	1490	1590		
Q_{24}	1457	3457	1567	3567	2368	e	
	3578	3678	1249	1459	4579		3–1–8
220128	2389	2489	3589	4589	3689		
	1240	1450	1260	2360	1470		5–2–7 4–6
2	1670	3670	2680	1290	2890		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_{25}	1256	2356	1456	3456	1257	e	
	1268	1468	3468	1278	1478		3–1–9, 2–4
220576	3478	3469	3569	3489	4789		
	1250	2560	1270	1780	3490		5–8, 6–7
2	2590	3590	2790	3790	7890		
Q_{26}	1356	2356	1367	2367	1458	e	
	2458	1568	2568	1678	2349		2–1 (9), 3–8
222144	2459	2369	2569	2379	4589		
	1370	1470	3670	1480	1580		4–6, 5–7
2	1780	2490	3790	4790	4890		
Q_{27}	3456	1257	2357	3457	1268	Z_2	
	1468	3468	1278	2378	1478	(1,2)(3,4)(5,6)(7,8)	1–3, 2–4
225664	3478	3459	3469	3579	4689		
	1270	2570	1280	1680	1290		5–8, 6–7
4	2590	3590	1690	4690	5690		
Q_{28}	1357	1367	1268	1368	2468	e	
	3468	2459	1379	1579	2579		1–4, 2–3
225808	1289	2489	2589	1689	1789		
	2450	3450	1360	2460	3460		5–6
4	3570	4570	3670	2480	5790		
Q_{29}	2347	2457	1258	1568	2478	Z_3	
	2578	1369	3469	3479	3679	(1,2,3)(4,6,5)(7,9,8)	1–4, 2–6
228528	1589	1689	1789	2789	3789		
	2340	1250	2450	1360	3460		3–5
4	1560	4560	3470	2580	1690		
Q_{30}	2345	2356	2457	2567	1678	e	
	5678	1349	2349	2359	2569		2–1–5
228568	1389	2389	1689	2689	6789		
	1340	2340	3450	1470	4570		3–7, 4–6
2	1670	5670	1480	1780	1890		
Q_{31}	3456	2358	3458	1468	3468	Z_2	
	1278	2378	1478	3478	2359	(1,2)(3,4)(5,6)(9,0)	3–1–5
231240	3459	3569	1279	2579	2789		
	1460	3460	4560	1270	1670		4–2–6
2	1780	1290	2590	1690	5690		
Q_{32}	2357	2457	2467	1358	1458	e	
	3578	4578	1369	2369	2379		2–1–7
233184	2479	2679	1389	2389	3589		
	1450	1460	2460	4570	4670		3–4, 5–6
2	1380	1580	1680	1690	2690		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_{33}	1267	1467	3467	3458	1468	e	
	3468	1259	1459	1269	1469		1—3 2—4
233352	1279	2589	3589	4589	4689		
	2350	1270	2370	2570	1670		5—6
4	3670	3480	3580	3780	2590		
Q_{34}	2357	1467	2467	1567	2567	Z_2	
	1358	2358	1468	1568	2568	(1,2)(3,4)(5,6)(7,8)(9,0)	1—2 3—6
233472	2359	2379	2479	2679	3589		
	1460	4670	1380	1480	1580		4—5 7—8
4	2390	1490	3490	4790	3890		
Q_{35}	2367	2567	1348	2348	1458	e	
	2458	2368	1568	2568	2378		1—2 3—5
234976	1459	1569	2569	5679	1489		
	1340	2370	3470	2670	3480		4—6
4	1490	1590	1790	3790	6790		
Q_{36}	1346	2346	1456	2456	2347	e	
	1457	2457	1368	1469	1569		1—2 3—5
236704	1579	2579	1589	1689	5789		
	2340	3460	2370	2570	2380		4—8 6—7
4	3680	2780	1890	3890	7890		
Q_{37}	4567	1238	2348	3478	3478	Z_3	
	4578	4678	1239	1259	1569	(1,2,3)(4,6,5)(8,0,9)	1—4
249640	2579	4579	5679	2389	2589		
	1230	1360	3460	1670	4670		2—6 3—5
4	5670	1380	3480	1290	1690		
Q_{38}	2346	2456	2347	2457	1368	e	
	3468	1568	4568	1578	4578		8—2—1—4
287872	2349	1579	2579	1689	5789		
	1360	2360	3460	2470	1580		3—5 6—7
	1390	2390	1790	2790	1890		
Q_{39}	2356	2367	3467	1258	2358	e	
	1458	2568	1278	1478	2678		3—1—6
300568	4678	2369	1479	4679	1589		
	1250	2350	3670	1480	1490		2—4 5—7
	3490	1590	3590	3690	4790		
Q_{40}	1235	1246	1256	1467	1567	e	
	2358	2568	3478	3578	4678		1—8 2—7
304288	5678	2359	1269	1469	4789		
	1250	4670	3580	3780	1290		3—6 4—5
	2390	1490	3490	4790	3890		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]					Type of aut(Q) and generators	Universal edges
Q_{41}	2345	2346	2357	1458	2458	Z_2	
	3458	1468	3468	1578	2578	(2,3)(4,5)(6,7)(9,0)	2–1–3
304512	1678	2359	1469	3469	1789		
	2340	1570	2570	1680	2390		4–7, 5–6
	1690	3690	1790	2790	6790		
Q_{42}	1235	1246	1256	2357	2567	Z_2	
	1468	1568	3478	3578	4678	(1,2)(3,4)(5,6)(7,8)	1–7, 2–8
308224	5678	2359	1469	3789	4789		
	1250	1260	3570	4680	1290		3–6, 4–5
	2390	1490	3490	3790	4890		
Q_{43}	1235	1246	1256	2357	2567	Z_2	
	1468	1568	3478	3578	4678	(1,2)(3,4)(5,6)(7,8)(9,0)	1–7, 2–8
309504	5678	2359	1269	3579	4789		
	1250	1460	4680	3780	1290		3–6, 4–5
	2390	1490	3490	3790	4890		
Q_{44}	2346	2456	2347	2457	1368	e	
	2368	3468	1378	3478	1578		2–1–4, 3–5
310432	4578	2459	1369	2369	1789		
	2460	1570	4570	1380	1590		6–7
	2590	1690	2690	5790	1890		
Q_{45}	2357	3457	2367	1467	3467	Z_2	
	2358	1458	3458	1468	3468	(1,2)(3,4)(5,6)(7,8)(9,0)	1–3, 2–4
311616	2359	1679	2679	4679	1489		
	1460	2370	1580	2580	3580		5–6, 7–8
	1290	2590	1690	2790	1890		
Q_{46}	2345	3456	2457	4567	1368	Z_2	
	3468	4578	4569	1279	2579	(2,3)(4,5)(6,7)(8,9)	4–1–5
312320	1689	4689	1789	5789	6789		
	1230	2340	2350	1360	3460		2–6, 3–7
	1270	2570	1280	1390	1890		
Q_{47}	2347	1467	3467	1567	2348	e	
	2358	1468	2468	1568	2568		2–1–3
313360	4678	1569	1479	3479	2389		
	2340	1670	1580	2580	2390		4–5
	1590	2590	3590	1790	3790		
Q_{48}	2457	2367	2467	1358	2358	e	
	1458	2458	1368	2368	2468		1–2, 3–4
325696	4578	1459	4579	2679	1389		
	1360	2360	2470	1580	1390		5–6
	1590	3690	1790	4790	6790		

TABLE 1 (continued)

det/100 see Table	Missing faces[1]						Type of aut(Q) and generators	Universal edges
Q_{49}	2356	2456	2357	2457	1368	e		
	1468	3568	4568	1378	2378			2 1 5
330504	3678	2459	1379	2379	1689			•—•—•
	1460	4560	2570	1380	1490			3 4
	2490	4590	1790	2790	1890			•—•
Q_{50}	1356	1367	2367	2467	1358	e		
	2458	3458	2378	2478	3578			1 2
340920	3678	1369	2479	1589	4589			•—•
	1350	1670	2670	2480	1490			
	2490	1590	4590	1690	2690			
Q_{51}	1357	1358	2358	2458	2368	e		
	2468	2678	1359	1459	2469			1 2 3 4
345856	4569	1479	4679	3589	4589			•—• •—•
	2460	1370	2370	1670	2670			
	4670	1380	2380	1590	1790			

[1] Vertex number 10 is represeted by 0 in Tables 1, 2, 3.

Third proof: For a universal edge E of Q_i, the quotient polytope Q_i/E should be a neighborly 4-polytope with 8 vertices. But for each i between 38 and 51, Q_i has a universal edge E_i such that Q_i/E_i is not equivalent to any of the three polytopes N_1^8, N_2^8, N_3^8 in [AS, Table 1]. To prove this it suffices to show that the determinant of the "facet-edge-valence matrix" of Q_i/E_i is not one of the numbers 0, 1592640 or 1756160 (see [AS, Table 1]). In fact, Q_i/E_i is the complex M described in [GS].

The last two proofs have the advantage that we need not bother how Q_i was obtained from P.

4. Realization of 26 cases

Our goal is to find realizations in $\mathbf{R}^6$ for the 37 remaining cases $Q_1, \ldots, Q_{37}$. In this section we shall prove that in 26 cases one can realize Q_i by adding a suitably chosen tenth vertex to *any* cyclic 6-polytope with 9 vertices.

Let K be a polytope. A *tower* $\mathcal{T}$ in K is a strictly increasing sequence $\phi \neq \Phi_1, \ldots, \Phi_k$ $(k \geqq 1)$ of proper faces of K. Define:

$$\mathcal{T}_j = \{F\colon F \supset \Phi_j, F \text{ is a facet of } P\}, \qquad 1 \leqq j \leqq k.$$

Define also:

$$\mathscr{C}(P, \mathscr{T}) = \mathscr{F}_1 \setminus (\mathscr{F}_2 \setminus (\cdots \setminus \mathscr{F}_k) \ldots).$$

Recall the following lemma ([SH2, lemma 4.4]):

LEMMA 14. *There exists a point x that lies exactly beyond $\mathscr{C} = \mathscr{C}(P, \mathscr{T})$ (i.e., x lies beyond every member of $\mathscr{C}$ and beneath all the other facets of P).*

PROOF. By induction on the height k of the tower $\mathscr{T}$. The assertion is obviously true for $k = 1$. Suppose $k > 1$, and assume the assertion holds for $k - 1$. Define $\mathscr{T}' = \{\Phi_2, \ldots, \Phi_k\}$. By the induction hypothesis, there is a point x' which lies exactly beyond $\mathscr{C}' = \mathscr{C}(P, \mathscr{T}')$. Choose a point p in relint Φ_1 and let $x = (1 + \varepsilon)p - \varepsilon x'$. If ε is positive and sufficiently small, then x lies exactly beyond $\mathscr{F}_1 \setminus \mathscr{C}'$, but $\mathscr{C}(P, \mathscr{T}) = \mathscr{F}_1 \setminus \mathscr{C}'$. ∎

We call the construction of Lemma 14 *sewing* through the tower $\mathscr{T}$. We say that the polytope $Q = \text{conv}(P \cup \{x\})$ *is obtained at x by sewing P through $\mathscr{T}$.*

26 cases Q_i are obtained by sewing P through some tower $\mathscr{T}$. The towers are listed in Table 2.

Take, for example, the case Q_5. Here, $\mathscr{T} = \{19, 149, 1459, 124589\}$.

$\mathscr{C} = \mathscr{C}(P, \mathscr{T}) = \{123459,\ 123569,\ 134569,\ 123679,\ 145679,\ 125689,\ 123789,\ 145789,\ 126789,\ 156789\}$.

The interior 3-faces of $\Delta(\mathscr{C})$ are: 1359 1269 1569 1279 1579 1689.

The 2-faces of P, not in $\Delta(\mathscr{C})$, are: 246 247 347 257 357 248 348 358 368 468.

Let Q be the polytope obtained by adding a tenth vertex exactly beyond $\mathscr{C}$. By Lemmas 2 and 3, the missing faces of Q are:

1357 1358 1368 1468 2468 1359 1269 2469 1569 1279 2479 1579 2579

3579 1689 2460 2470 3470 2570 3570 2480 3480 3580 3680 4680.

Here, as in Table 2, 0 represents the tenth vertex. The permutation (1, 7, 2, 9, 5, 3, 0) (6, 8) is a combinatorial equivalence between Q and Q_5.

5. Realization of 2 cases

Two more cases, Q_9 and Q_{15}, are obtained from a cyclic 6-polytope P with 9 vertices by sewing it twice and then omitting a vertex (Table 3). For example, start with the polytope Q_4, as described in Table 1. Q_4 is obtained from P by sewing (see Table 2). Sew Q_4 at the vertex * through the tower {16, 1680, 156780}. Denote the obtained polytope by K. K is neighborly, as the reader can easily check. Now omit the vertex 8, find the missing faces for the resulting

TABLE 2
Sewn polytopes

	Tower in P				Permutation
Q_1	19	1289	123789		identity
Q_2	19	1289	123489		(2,5,3,8,7,9,6,4,0)
Q_3	19	1459	134569		(1,5,0)(2,7,4,6)
Q_4	19	1289	124589		(1,4,0,2,7,9)(3,8,6)
Q_5	19	149	1459	124589	(1,7,2,9,5,3,0)(6,8)
Q_7	19	1239	123459		(1,3,8,9,5,6,2,0)(4,7)
Q_8	19	139	1342	134569	(1,8,9,6,7,3,5,4)(2,0)
Q_{10}	19	1239	123789		(2,5,3,9,8,0)(4,6)
Q_{11}	19	179	1789	145789	(1,3,2,0)(6,8,7)
Q_{12}	19	179	1789	134789	(1,3,5,2,9,7,6,8,0)
Q_{13}	19	1239	123679		(1,3,0)(2,9,5)(4,6)
Q_{14}	19	1239	123569		(2,8,9,6,4,7,3,0)
Q_{16}	19	159	1569	156789	(1,7,6,2,5,9,4,8,0)
Q_{17}	19	149	1459	145679	(1,6,4,3,0,2,8,5)
Q_{18}	19	139	1349	134679	(1,3,8,9,5,4,6,7,2,0)
Q_{19}	19	169	1679	123679	(1,5,8,0)(2,9,4)(3,6)
Q_{20}	19	149	1459	145789	(1,5,4,3,9,2,8,0)
Q_{21}	19	1349	134569		(1,5,6,7,2,9,3,0)
Q_{22}	19	1459	145679		(1,2,5,9,8,6,4,0)(3,7)
Q_{23}	19	1459	124589		(1,3,5,0)(2,8,7)(4,9,6)
Q_{24}	19	1459	123459		(1,3,6,5,0)(2,9,8,7)
Q_{25}	19	169	1679	134679	(1,9,3,7,4,5,8,0)(2,6)
Q_{26}	19	159	1569	123569	(1,2,0)(3,5,8,6)(4,7)
Q_{30}	19	1349	134679		(1,5,4,8,7,3,0)(2,9)
Q_{31}	19	1349	134789		(1,5,6,2,8,9,3,0)(4,7)
Q_{32}	19	1459	145789		(1,2,5,0)(3,6)(4,8,9,7)

TABLE 3
Two cases

Polytope	Is obtained from	By sewing at * through	Omitting	And renaming the vertices by
Q_9	Q_4	16, 1680, 156780	8	$\begin{pmatrix}1&2&3&4&5&6&7&9&0&*\\8&5&3&1&9&6&4&7&0&2\end{pmatrix}$
Q_{15}	Q_{30}	46, 4690, 246790	0	$\begin{pmatrix}1&2&3&4&5&6&7&8&9&*\\1&5&8&4&3&6&7&0&9&2\end{pmatrix}$

polytope by the rule of Lemma 4, and rename the vertices according to the function

$$\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 9, & 0, & * \\ 8, & 5, & 3, & 1, & 9, & 6, & 4, & 7, & 0, & 2 \end{pmatrix}.$$

The resulting polytope is Q_9.

Each of the 28 polytopes Q_i covered by Tables 2 and 3 has at least one pair of adjacent universal edges. The remaining nine spheres have none, and we can prove that the techniques of sewing and omitting vertices are not sufficient to establish their polytopality.

6. Realization of the remaining 9 cases

These cases were decided by the first author, using computational rather than combinatorial methods.

Let Q be one of the 9 remaining types. Now we identity Q with its set of facets $\{F_1, \dots, F_{50}\}$, where each facet F_i is a 6-subset of $\{1, \dots, 10\}$. Regard an indexed set $W = \{x_1, \dots, x_{10}\}$ of ten points in $\mathbf{R}^6$. Assume that $x_i = (x_{i,1}, \dots, x_{i,6})$, $i = 1, \dots, 10$, define $x_{i,0} = 1$ for $i = 1, \dots, 10$ and denote by X the 10×7 matrix (x_{ij}). For a sequence $I = (i_0, \dots, i_6)$ of 7 distinct numbers in $\{1, \dots, 10\}$, denote by $X(I)$ the 7×7 submatrix $(x_{i_k,j})$, $0 \leqq k, j \leqq 6$. For each face $F = \{i_1, \dots, i_6\}$ in Q, consider the requirement

(F) $\quad \operatorname{sg}\det X(a, i_1, \dots, i_6) = \operatorname{sg}\det X(b, i_1, \dots, i_6) \neq 0$ for all $a, b \in \{1, \dots, 10\} \setminus F$.

Note that X satisfies (F) if and only if $\dim(\operatorname{conv} W) = 6$ and $\operatorname{conv}\{x_i : i \in F\}$ is a facet of $\operatorname{conv} W$. It follows easily that if X satisfies (F) for all $F \in Q$, then $\operatorname{conv} W$ is a realization of Q. Note that all the requirements (F) together determine the signs of all determinants that appear in them up to a common reversal of all signs. This follows from the connectivity of the incidence graph of facets and subfacets of Q. (The common sign reversal corresponds, e.g., to a reflection of the points x_i in a hyperplane.)

Thus we have reduced the realization problem of Q to that of solving a suitable system of determinantal (strict) inequalities in the variables $x_{i,j}$ ($1 \leqq i \leqq 10$, $1 \leqq j \leqq 6$). Since the vertices of a neighborly polytope must be in general position, we can choose $x_1, \dots, x_7$ to be the origin and the unit vectors $e_1, \dots, e_6$. This reduces the number of variables to 18, and the size of the determinants to 1×1, 2×2 and 3×3.

Then we observe that some of our inequalities are implied by the others, due to the Plücker–Grassmann relations for determinants [HP]. This leads to a substantial reduction in the number of inequalities. The realizations listed in Table 4 were obtained by solving this reduced system.

REMARKS. The combinatorial structure of a neighborly polytope determines the structure of all its subpolytopes (Lemma 4). This implies that the requirements (F) ($F \in Q$) indirectly determine the signs of all the 7×7 subdeterminants of X (up to a common sign reversal).

The first author used the main idea which leads to affine Gale-diagrams, see [R1], thus translating the problem of finding 10 points in $\mathbf{R}^6$ with preassigned orientations of all 7-tuples to that of finding 10 points in $\mathbf{R}^2$ with preassigned orientations of 3-tuples. This reduces our realization problem to a more tractable geometric problem in the plane. He solved the planar problem successfully. For further details, see [BoSt], [BoSt3], [St].

7. Concluding remarks

(1) Seventeen of the 37 polytopes Q_i have non-trivial combinatorial automorphisms (see Table 1). The cyclic polytope Q_1 has a symmetric realization, i.e., a realization in which all the combinatorial automorphisms are induced by isometries. (Take the convex hull of 10 evenly spaced points on the trigonometric moment curve in $\mathbf{R}^6$.) In [BEK] an example is given of a simplicial 4-polytope with 10 vertices that admits no symmetric realization. The vertices of that polytope are necessarily not in general position. It would be interesting to know whether our polytopes Q_i do have symmetric realizations.

(2) Call a simplicial d-polytope P *k-stacked* if P has a triangulation with no additional vertices and no interior cells of dimension less than $d-k$. A 1-stacked polytope is just an ordinary stacked polytope. It is easy to see that a neighborly $2k$-polytope is k-stacked. It can be shown that all the vertex figures of a neighborly $2m$-polytope with v vertices are $(m-1)$-stacked and $(m-1)$-neighborly $(2m-1)$-polytopes with $v-1$ vertices. It can also be proved that for $1 \leqq k \leqq [d/2]$ there is only one combinatorial type of k-stacked k-neighborly d-polytope with $d+3$ vertices. Hence, the first interesting family of k-stacked k-neighborly d-polytopes is the family of 2-stacked 2-neighborly 5-polytopes with 9 vertices. This family includes, e.g., the vertex figures of our 37 polytopes Q_i. Some k-stacked k-neighborly d-polytope can be obtained by "splitting" a vertex of a neighborly $2k$-polytope (see [AP, section 6] and [MW]). It can be shown that if x is a vertex of a neighborly $2m$-polytope Q, then the vertex figure of Q at x is obtained by "splitting" if and only if x lies on two universal edges of Q. It would be interesting to know if there exist 2-stacked 2-neighborly

TABLE 4
Nine cases

	Vertex	Coordinates					
	1	0	0	0	0	0	0
	2	1	0	0	0	0	0
	3	0	1	0	0	0	0
	4	0	0	1	0	0	0
	5	0	0	0	1	0	0
	6	0	0	0	0	1	0
	7	0	0	0	0	0	1
	8	0.001	−0.0009	0.001	1.0005	−0.002	0.001
Q_6	9	−2	−1.6	2	1.6	−1.5	1
	10	2	1	−1	0.1	−1.1	1
	8	0.01	0.25	−0.01	0.96	0.03	0.02
Q_{27}	9	1	−0.5	−0.5	0.5	−0.5	0.5
	10	−0.32	1.29	0.65	0.32	0.033	−0.003
	8	−0.165	0.174	−0.165	0.331	0.165	0.496
Q_{28}	9	−0.4	0.5	−0.2	−0.5	0.6	0.2
	10	−0.164	0.18	−0.246	−0.164	0.164	0.328
	8	0.1408	1.831	0.8592	−0.5634	−0.2817	0.7042
Q_{29}	9	4	−1	1	1	−3	1
	10	0.8547	0.8547	0.4188	−0.2735	−0.8547	0.8547
	8	2	−20	−1	−1	1	2.19
Q_{33}	9	−0.172	0.344	0.344	−0.172	0.344	0.344
	10	−0.4	−0.2	1	−0.04	0.4	0.02
	8	0.0001	−0.0001	0.0001	−0.0001	0.0001	1
Q_{34}	9	−0.167	−0.5	0.167	−0.01	1	0.26
	10	0.5	0.5	−0.75	0.8125	−0.01	0.3025
	8	0.81	−0.086	−0.931	0.069	0.672	1.41
Q_{35}	9	1.18	1.18	−0.235	−0.706	0.059	0.706
	10	1.32	−4.6	−1.68	4.32	2.4	0.24
	8	1.32	−1	0.982	1	−0.992	1
Q_{36}	9	−0.25	1.25	0.251	−0.5	0.249	−0.25
	10	−2	1	0.498	−0.5	1.5	−1.5
	8	−0.251	−0.209	0.0418	0.418	0.836	0.209
Q_{37}	9	−0.225	−0.115	0.449	0.225	0.674	0.225
	10	−0.0461	−0.395	0.132	1.050	0.132	0.263

5-polytopes that are not vertex figures of neighborly 6-polytopes. Possible candidates would be some of the "vertex figures" of the non-polytopes Q_{38}–Q_{51}.

(3) We would like to draw the reader's attention to the following unexplained phenomenon. The determinants $\det Q_i$ separate the polytopes from the non-polytopes: $\det Q_i \leqq 249640$ for $i \leqq 37$, while $\det Q_i \geqq 287872$ for $i \geqq 38$! A similar phenomenon was discovered by Altshuler [AL] while enumerating the 2-neighborly combinatorial 3-manifolds with 10 vertices.

References

[AL] A. Altshuler, *Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with ten vertices*, Can. J. Math. **29** (1977), 400–420.

[AM] A. Altshuler and P. McMullen, *The number of simplicial neighborly d-polytopes with d + 3 vertices*, Mathematika **20** (1973), 263–266.

[AP] A. Altshuler and M. A. Perles, *Quotient polytopes of cyclic polytopes. Part* 1, Isr. J. Math. **36** (1980), 97–125.

[AS] A. Altshuler and L. Steinberg, *Neighborly 4-polytopes with 9 vertices*, J. Comb. Theory A **15** (1973), 270–287.

[BR] D. Barnette, *A family of neighborly polytopes*, Isr. J. Math. **39** (1981), 127–140.

[BEK] J. Bokowski, G. Ewald and P. Kleinschmidt, *On combinatorial and affine automorphisms of polytopes*, Isr. J. Math. **47** (1984), 123–130.

[BL] L. J. Billera and C. W. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Comb. Theory A **31** (1981), 237–255.

[BM] H. Bruggesser and P. Mani, *Shellable decompositions of cells and spheres*, Math. Scand. **29** (1971), 197–205.

[BoSt1] J. Bokowski and B. Sturmfels, *On the coordinatization of oriented matroids*, Discrete and Computational Geometry **1** (1986), 293–306.

[BoSt2] J. Bokowski and B. Sturmfels, *Polytopal and nonpolytopal spheres: an algorithmic approach*, Isr. J. Math. **57** (1987), 257–271.

[BoSt3] J. Bokowski and B. Sturmfels, *An infinite family of minor-minimal nonrealizable 3-chirotopes*, manuscript, 1986.

[DK1] G. Danaraj and V. Klee, *Shelling of spheres and polytopes*, Duke Math. J. **41** (1974), 443–451.

[DK2] G. Danaraj and V. Klee, *Which spheres are shellable?*, Ann. Discrete Math. **2** (1978), 33–52.

[GR] B. Grünbaum, *Convex Polytopes*, Interscience, London/New York/Sydney, 1967.

[GS] B. Grünbaum and V. P. Sreedharan, *An enumeration of simplicial 4-polytopes with 8 vertices*, J. Comb. Theory **2** (1967), 437–465.

[HP] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Cambridge, 1968.

[MS] P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, London Math. Soc. Lecture Note Series 3, Cambridge University Press, 1971.

[MW] P. McMullen and D. W. Walkup, *a generalized lower-bound conjecture for simplicial polytopes*, Mathematika **18** (1971), 264–273.

[R1] M. Rieger, *Shemer-Sphären mit 10 Ecken im* $\mathbf{R}^6$, *Rückführung auf ein ebenes Problem*, Staatsexamensarbeit, Bochum, 1983.

[SH1] I. Shemer, *Neighborly polytopes*, M.Sc. thesis, 1971 (in Hebrew).

[SH2] I. Shemer, *Neighborly polytopes*, Isr. J. Math. **43** (1982), 291–314.

[St] B. Sturmfels, *Affine Gale-diagrams*, manuscript, 1986.